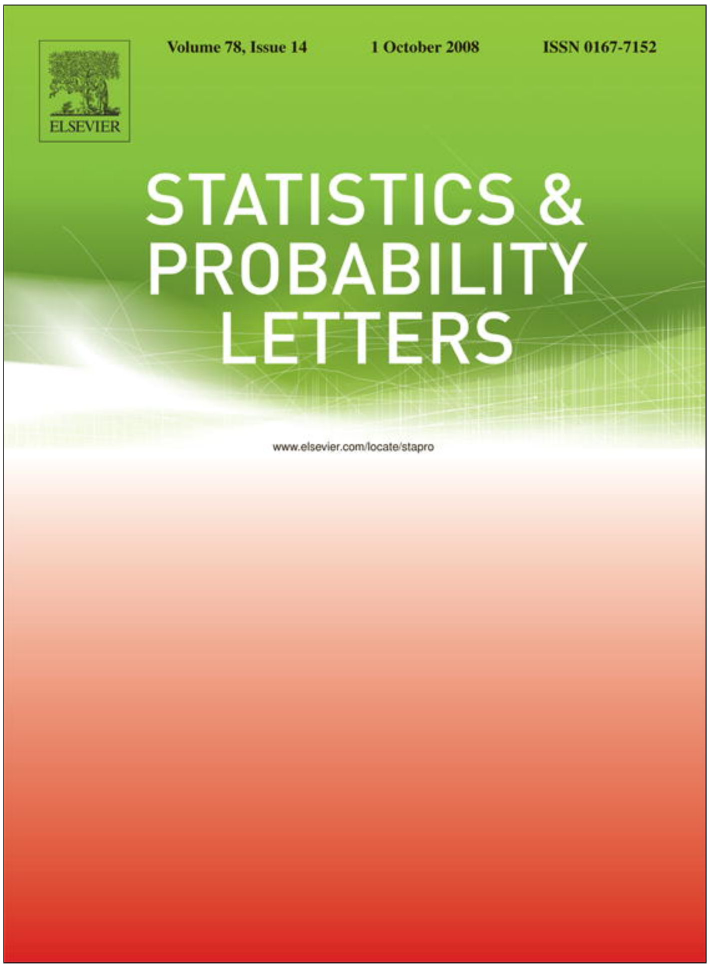
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The disk-percolation model on graphs

E. Lebensztayn*, P.M. Rodríguez

Statistics Department, Institute of Mathematics and Statistics, University of São Paulo, Rua do Matão 1010, CEP 05508-090, São Paulo, SP, Brazil

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Abstract

We study a long-range percolation model whose dynamics describe the spreading of an infection on an infinite graph. We obtain a sufficient condition for phase transition and prove an upper bound for the critical parameter of spherically symmetric trees.

MSC: 60K35; 60J80; 05C05; 82B26

1. Introduction

Spread-out and long-range percolation models have been studied extensively in the past few years. Generally, each pair of points in a (random) set is connected according to some probabilistic rule, and the main interest is the existence of a nontrivial critical phenomenon. Burton and Meester (1993) study the phase transition for a long-range independent percolation model on a stationary point process in $\mathbb{R}^d$ in which any two points are connected with a probability that decays exponentially in the distance between them. Other continuum models with different connection rules are considered in Häggström and Meester (1996) and Franceschetti et al. (2005). Penrose (1993) considers X equal to $\mathbb{Z}^d$ or a Poisson process in $\mathbb{R}^d$ with density 1, and studies the spread-out percolation model in which each pair of points of X within distance r is independently joined with probability p. See also Bollobás et al. (2007) for more results on this model. On the other hand, Alves et al. (2002) and Lebensztayn et al. (2005, 2006) study the frog model, a long-range percolation model on an infinite graph in which each vertex is connected to all the vertices visited by a random walk with random lifetime.

In this paper, we study a long-range percolation model on infinite graphs that we call the disk-percolation model. Its dynamics describes the spreading of an infection on the graph in the following way. We assign a random radius of infection R_v to each vertex v of an infinite, locally finite, connected graph $\mathcal{G}$, so that all the assigned radii are independent and identically distributed random variables with geometric distribution with parameter $(1 - p)$. Then, we define a growing process on $\mathcal{G}$ according to the following rules: (1) at time zero, only the root (a fixed vertex of $\mathcal{G}$) is declared infected, (2) at time $n \geq 1$, a new vertex is infected if it is at graph distance at most R_v of some vertex v previously infected, and (3) infected vertices remain infected forever. The principal purpose is to investigate the critical value $p_c(\mathcal{G})$ above which this epidemic process spreads indefinitely through the graph with positive probability.

* Corresponding author.
E-mail addresses: elcio@ime.usp.br (E. Lebensztayn), pablor@ime.usp.br (P.M. Rodríguez).

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The main results of the paper are presented in Section 2. First, we give a condition on the graph $\mathcal{G}$ under which the critical phenomenon is nontrivial, in the sense that both the subcritical and the supercritical phases are nonempty. In particular, we establish the presence of phase transition for a wide class of graphs. Secondly, we restrict our study to the case of spherically symmetric trees, and prove an upper bound for the critical parameter. This upper bound is a function of a constant presented in Lyons and Peres (2005) which is closely related to the Hausdorff dimension of the boundary of the tree. From this result, we obtain the asymptotic behavior of the critical parameter of the homogeneous trees. Section 3 is dedicated to the proofs.

Before we formally define the model, let us recall some graph terminology. We consider $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ an infinite, locally finite, connected graph, where $\mathcal{V}$ is the set of vertices and $\mathcal{E} \subset \{\{u, v\} : u, v \in \mathcal{V}, u \neq v\}$ is the set of edges of $\mathcal{G}$. Both $\mathcal{V}$ and $\mathcal{E}$ are countably infinite. If $\{u, v\} \in \mathcal{E}$, we say that u and v are neighbors, which is denoted by $u \sim v$. The degree of a vertex v is the number, $d(v)$, of its neighbors. The graph is locally finite if $d(v)$ is finite for every vertex v. If $\Delta := \sup\{d(v) : v \in \mathcal{V}\} < \infty$, the graph is said to be of bounded degree, and Δ is its maximal degree. A path of $\mathcal{G}$ is a finite sequence $v_0, v_1, \ldots, v_n$ of distinct vertices in which $v_i \sim v_{i+1}$ for each i. The graph is connected if there is a path between every pair of its vertices. A tree is a graph such that for each pair of vertices there is a unique path which connects them. Let $\text{dist}(u, v)$ denote the usual graph distance between vertices u and v, that is, $\text{dist}(u, u) = 0$ and, for $u \neq v$, $\text{dist}(u, v)$ is the minimal number of edges in a path connecting them. Let $\mathbf{0} \in \mathcal{V}$ be a fixed vertex of $\mathcal{G}$, that we call the root of $\mathcal{G}$, and for each $v \in \mathcal{V}$ define $|v| := \text{dist}(\mathbf{0}, v)$. For $d \geq 1$, denote by $\mathbb{Z}^d$ the d-dimensional hypercubic lattice and by $\mathbb{T}_d$ the homogeneous tree of degree $(d + 1)$.

Now we define the model in a formal way. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an infinite, locally finite, connected graph, and $\{R_v; v \in \mathcal{V}\}$ be a set of independent and identically distributed random variables, with distribution given by

$$\mathbf{P}(R_v = k) = (1 - p)\, p^k, \quad k = 0, 1, \ldots,$$

where $p \in [0, 1]$ is a fixed parameter. For each $v \in \mathcal{V}$, we define

$$B_v = \{u \in \mathcal{V} : \text{dist}(v, u) \leq R_v\},$$

and then we consider the nondecreasing sequence of random sets $I_0 \subset I_1 \subset \cdots$ defined inductively by

$$I_0 = \{\mathbf{0}\},$$
$$I_{n+1} = \bigcup_{v \in I_n} B_v, \quad n \geq 0.$$

Let $\mathcal{C} = \bigcup_{n \geq 0} I_n$.

Thus, R_v denotes the radius of infection of the vertex v, I_n is the set of infected vertices at time n, and $\mathcal{C}$ stands for the set of infected vertices of the realization. We call this model the *disk-percolation model* on $\mathcal{G}$ with parameter p, and denote it by $\mathcal{M}(\mathcal{G}, p)$.

Definition 1. A particular realization of the disk-percolation model *survives* if $|\mathcal{C}| = \infty$. Otherwise, we say that the realization *dies out*.

Hence, a realization dies out if and only if $I_n = I_{n+1}$ for some n. Since the survival probability is nondecreasing in p, we define the *critical parameter*

$$p_c(\mathcal{G}) = \inf\{p : \mathbf{P}(\mathcal{M}(\mathcal{G}, p) \text{ survives}) > 0\}.$$

As usual, we say that the disk-percolation model on $\mathcal{G}$ exhibits *phase transition* if $0 < p_c(\mathcal{G}) < 1$.

It is essential to underline that we are dealing with a directed percolation model. For $u, v \in \mathcal{V}$, we define

$$\{u \to v\} = \{v \in B_u\} \quad \text{and} \quad \{u \nrightarrow v\} = \{v \notin B_u\}.$$

This definition clarifies that the disk-percolation model is a *long-range, dependent, directed percolation model* on the directed graph with vertex-set $\mathcal{V}$ and edge-set $\mathcal{V} \times \mathcal{V}$. For $u, v \in \mathcal{V}$,

$$\mathbf{P}(u \to v) = \mathbf{P}(R_u \geq \text{dist}(u, v)) = p^{\text{dist}(u,v)}. \tag{1}$$

In addition, notice that our model survives if and only if there exists an infinite sequence of distinct vertices $\mathbf{0} = v_0, v_1, \ldots,$ such that $v_j \to v_{j+1}$ for all $j \geq 0$.

2. Main results

2.1. Sufficient condition for phase transition

We state nontrivial bounds for the critical parameter. Recall that $\mathcal{G}$ is an infinite, locally finite and connected graph. First, by a direct coupling with a site percolation model,

Proposition 2. *Let $p_c^{site}(\mathcal{G})$ denote the critical probability of the independent site percolation model on $\mathcal{G}$. It holds that $p_c(\mathcal{G}) \leq p_c^{site}(\mathcal{G})$.*

For a general reference on percolation, see Grimmett (1999). Next, by comparing our model to a suitable Galton–Watson branching process, we obtain a lower bound for the critical parameter on graphs of bounded degree.

Proposition 3. *Suppose that $\mathcal{G}$ is a graph of maximal degree $\Delta < \infty$. Then,*

$$p_c(\mathcal{G}) \geq -1 + \left(1 + \frac{1}{\Delta - 1}\right)^{1/2}.$$

Observe that the fact that $\mathcal{G}$ is connected implies $\Delta > 1$. The details of both proofs are given in Section 3.1.

Therefore,

Theorem 4. *Let $\mathcal{G}$ be a graph of bounded degree such that $p_c^{site}(\mathcal{G}) < 1$. Then,*

$$0 < p_c(\mathcal{G}) < 1.$$

Consequently, the disk-percolation model exhibits phase transition on $\mathbb{Z}^d$, $d \geq 2$, on trees of bounded degree with branching number greater than 1, as well as on nonamenable graphs of bounded degree. See Lyons and Peres (2005) for more details about percolation models on these graphs.

Remark 5. As an example of a graph with absence of phase transition, we cite $\mathbb{Z}$, for which the critical parameter equals 1. To prove this fact, one can use the argument used in Theorem 2.1 of Lebensztayn et al. (in press). Another example is the factorial tree, for which the critical parameter equals 0 (see Example 8(3) and Theorem 9).

2.2. Spherically symmetric trees

Here we present the main result of the paper, which establishes another upper bound for the critical parameter of the disk-percolation model on spherically symmetric trees. Before this, we need a few definitions.

Let $T = (\mathcal{V}, \mathcal{E})$ be an infinite, locally finite tree, with root $\mathbf{0}$. Recall that $|v| = \text{dist}(\mathbf{0}, v)$ for each $v \in \mathcal{V}$. We consider the usual partial order on $\mathcal{V}$: for $u, v \in \mathcal{V}$, we say that $u \leq v$ if u is one of the vertices of the path connecting $\mathbf{0}$ and v; $u < v$ if $u \leq v$ and $u \neq v$. We call v a *descendant* of u if $u \leq v$ and denote by $T^u = \{v \in \mathcal{V} : u \leq v\}$ the set of descendants of u; v is said to be a *successor* of u if $u \leq v$ and $u \sim v$. For $u \in \mathcal{V}$ and $n \geq 1$, we define

$$T_n^u = \{v \in T^u : |v| \leq |u| + n\},$$
$$\partial T_n^u = \{v \in T^u : |v| = |u| + n\},$$
$$M_n(u) = |\partial T_n^u|.$$

Definition 6. For an infinite, locally finite tree $T = (\mathcal{V}, \mathcal{E})$, we define

$$\dim\inf \partial T := \lim_{n\to\infty} \min_{v \in \mathcal{V}} \frac{1}{n} \log M_n(v). \tag{2}$$

Remark 7. The limit in (2) is easily seen to exist in $[0, \infty]$ by a standard superadditive argument (Fekete's Lemma). More details on this constant can be found in Section 13.5 of Lyons and Peres (2005), which is based on some unpublished ideas of H. Furstenberg. In particular, the authors prove that for an infinite tree $T = (\mathcal{V}, \mathcal{E})$ of bounded degree,

$$\dim\inf\partial T = \inf\{\dim\partial T^* : T^* \in \mathcal{D}(T)\},$$

where $\dim\partial T^*$ denotes the Hausdorff dimension of the boundary of the tree T^* and $\mathcal{D}(T)$ is the closure of the set of the descendant trees of T, $\{T^v : v \in \mathcal{V}\}$.

Now we fix our attention on spherically symmetric trees. A *spherically symmetric tree* is a tree obtained by a sequence of positive integers $f_1, f_2, \ldots$ as follows. Start with the root $\mathbf{0}$, which has f_1 successors. Then each one of these f_1 successors has f_2 successors, and so on. In other words, a tree is spherically symmetric if the degree of any vertex depends only on its distance from the root. An example is the homogeneous tree $\mathbb{T}_d$, for which $f_1 = d + 1$ and $f_i = d$ for all $i \geq 2$.

Example 8. We present the value of $\dim\inf\partial T$ for some spherically symmetric trees:

1. For $d \geq 1$, $\dim\inf\partial\mathbb{T}_d = \log d$.
2. If T is a spherically symmetric tree such that f_i equals a or b according to i being odd or even, then $\dim\inf\partial T = \log\sqrt{ab}$.
3. Let $T_!$ be the factorial tree, that is, the spherically symmetric tree for which $f_i = i + 1$ for all $i \geq 1$. Then $\dim\inf\partial T_! = \infty$.

The main result of the paper provides an upper bound for the critical parameter of a spherically symmetric tree.

Theorem 9. *For any spherically symmetric tree T,*

$$p_c(T) \leq 1 - \left(1 - \mathrm{e}^{-\dim\inf\partial T}\right)^{1/2}.$$

The proof of Theorem 9 is presented in Section 3.2. Briefly, the argument consists in coupling a sequence of branching processes whose survival implies the survival of the disk-percolation model. This technique is used in Lebensztayn et al. (2005, 2006) for the frog model and a self-avoiding random walks model on homogeneous trees, respectively.

Notice that if $\dim\inf\partial T = \dim\partial T$ (Hausdorff dimension of ∂T), then the upper bound given in Theorem 9 is better than the one given in Proposition 2, namely, $\mathrm{e}^{-\dim\partial T}$ (see Theorem 6.2 of Lyons, 1990, or Theorem 4.16 of Lyons and Peres, 2005). The usefulness of this improvement becomes clear in the case of the homogeneous trees, for which Proposition 3, Theorem 9 and Example 8(1) yield the following result.

Theorem 10. *For any $d \geq 2$,*

$$-1 + \left(1 + \frac{1}{d}\right)^{1/2} \leq p_c(\mathbb{T}_d) \leq 1 - \left(1 - \frac{1}{d}\right)^{1/2}.$$

As a consequence, we obtain the asymptotic behavior of $p_c(\mathbb{T}_d)$.

Corollary 11. $p_c(\mathbb{T}_d) = 1/(2d) + O(1/d^2)$ *as* $d \to \infty$.

3. Proofs

3.1. Proofs of Propositions 2 and 3

Proof of Proposition 2. We consider the natural coupling of the disk-percolation model and the independent site percolation model on $\mathcal{G}$: given a realization of $\mathcal{M}(\mathcal{G}, p)$, we say that a vertex v is open if $R_v \geq 1$, and closed otherwise. This coupling is such that the occurrence of site percolation implies the survival of the disk-percolation model. This fact yields the desired conclusion. □

Proof of Proposition 3. Fix any vertex $v \neq \mathbf{0}$ and define the random variable

$$\begin{aligned} X_v &= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}\Delta(\Delta-1)^{j-1}\right)\mathcal{I}_{\{R_v=i\}} - \mathcal{I}_{\{R_v\geq 1\}} \\ &= \sum_{j=1}^{\infty}\Delta(\Delta-1)^{j-1}\,\mathcal{I}_{\{R_v\geq j\}} - \mathcal{I}_{\{R_v\geq 1\}}, \qquad (3) \end{aligned}$$

where $\mathcal{I}$ denotes the indicator function of the event in the subscript. Consider the Galton–Watson branching process that starts with a number of individuals distributed as $|B_{\mathbf{0}}|$ and whose family size is distributed as X_v. That is, the 0th generation has $|B_{\mathbf{0}}|$ individuals and the individuals in successive generations generate offspring independently according to the law of X_v. Observe that the disk-percolation model on $\mathcal{G}$ dies out if this branching process becomes extinct. To see this, recall that in the disk-percolation model each infected vertex v transmits the infection to all the vertices in B_v. The domination occurs because for every vertex v the summation in Eq. (3) is an upper bound for the number of vertices in $B_v \setminus \{v\}$. For $v \neq \mathbf{0}$, we subtract 1 from this summation if $R_v \geq 1$ due to the existence of a vertex neighbor to v which has been previously infected.

Notice now that $\mathbf{P}(|B_{\mathbf{0}}| < \infty) = 1$ for every $p < 1/(\Delta - 1)$, since in this case,

$$\mathbf{E}(|B_{\mathbf{0}}|) \leq 1 + p\,\Delta \sum_{j=1}^{\infty} (p(\Delta - 1))^{j-1} < \infty.$$

Consequently, if $p < 1/(\Delta - 1)$ and

$$\mathbf{E}(X_v) = \frac{p\,\Delta}{1 - p(\Delta - 1)} - p < 1,$$

then $\mathcal{M}(\mathcal{G}, p)$ dies out with probability 1. The stated lower bound follows by an elementary computation. □

3.2. *Proof of Theorem 9*

Here is the main idea: For each n, we define a Galton–Watson branching process in a varying environment whose survival implies the survival in our model. Obtaining a sufficient condition for the supercritical behavior of each one of these branching processes leads us to the desired result. As already mentioned, this technique of using embedded branching processes is used for random walks models in Lebensztayn et al. (2005, 2006). It is also very similar to some arguments used for the contact process on a homogeneous tree — see for instance the paper by Lalley and Sellke (1998).

Let us start with a few definitions.

Definition 12. For $u \in \mathcal{V}$ and $v \in \partial T_n^u$, consider $v_0 = u < v_1 < \cdots < v_{n-1} < v_n = v$ the path connecting u and v. We denote by $\{u \overset{c}{\to} v\}$ the event that $\{u \to v\}$ or there exists $1 \leq i_1 < \cdots < i_k \leq n - 1$ such that

$$\{u \to v_{i_1}\} \bigcap \left(\bigcap_{j=1}^{k-1} \{v_{i_j} \to v_{i_{j+1}}\} \right) \bigcap \{v_{i_k} \to v\}.$$

We denote the complement of this event by $\{u \overset{c}{\nrightarrow} v\}$.

Now fix an integer $n \geq 1$. Define $\mathcal{Z}_0^n = \{\mathbf{0}\}$, and for $j = 1, 2, \ldots$ define

$$\mathcal{Z}_j^n = \bigcup_{u \in \mathcal{Z}_{j-1}^n} \{v \in \partial T_n^u : u \overset{c}{\to} v\}.$$

For $j = 0, 1, \ldots$ let $Z_j^n = |\mathcal{Z}_j^n|$ be the cardinality of $\mathcal{Z}_j^n$.

Proposition 13. *For every fixed $n \geq 1$, $(Z_j^n)_{j \geq 0}$ is a Galton–Watson branching process in a varying environment whose survival implies the survival in the disk-percolation model. In addition, for $j \geq 0$, the mean offspring number of an individual of the jth generation satisfies*

$$\mu_j^n \geq M_n(v)\, G_n(p), \tag{4}$$

where $|v| = jn$, and

$$G_n(p) = p^n (2 - p)^{n-1}. \tag{5}$$

The Galton–Watson branching process in a varying environment is the generalization of the standard branching process in which the offspring distribution depends on the generation number. We refer to D'Souza and Biggins (1992) for more details about these processes. Since the first statement in Proposition 13 is clear (by construction), its proof follows from the next two lemmas. To understand (4), observe that every vertex v of the jth generation is at distance jn from the root and each one of the $M_n(v)$ vertices in ∂T_n^v is a child of v with probability $F_n(p)$.

Lemma 14. *With the notations introduced in Definition 12, we have that* $\mathbf{P}(v_0 \overset{c}{\to} v_n) = F_n(p)$*, where* F_n *is inductively given by*

$$F_n(p) = p^n \prod_{k=1}^{n-1}(1-p^k) + \sum_{j=1}^{n-1} p^{n-j}\, F_j(p) \prod_{k=1}^{n-j-1}(1-p^k).$$

As usual, an empty product equals 1 *and an empty sum is* 0*.*

Proof. Let $n \geq 2$ and for each $1 \leq j \leq n-1$ denote by $\{v_0 \dashv v_j\}$ the event $\{v_0 \to v_j, v_0 \not\to v_{j+1}\}$. Observe that

$$\mathbf{P}(v_0 \overset{c}{\to} v_n) = \mathbf{P}(v_0 \to v_n) + \sum_{j=1}^{n-1} \mathbf{P}(v_0 \dashv v_j)\, \mathbf{P}\left(\bigcup_{k=1}^{j}\{v_k \overset{c}{\to} v_n\}\right),$$

with $\mathbf{P}(v_0 \dashv v_j) = \mathbf{P}(v_0 \to v_j) - \mathbf{P}(v_0 \to v_{j+1})$.

Thus, pulling out common factors, we obtain

$$\begin{aligned}\mathbf{P}(v_0 \overset{c}{\to} v_n) &= \mathbf{P}(v_0 \to v_n)\, \mathbf{P}(v_1 \overset{c}{\not\to} v_n, \ldots, v_{n-1} \overset{c}{\not\to} v_n)\\ &\quad + \sum_{j=1}^{n-2} \mathbf{P}(v_0 \to v_{n-j})\, \mathbf{P}(v_1 \overset{c}{\not\to} v_n, \ldots, v_{n-j-1} \overset{c}{\not\to} v_n, v_{n-j} \overset{c}{\to} v_n)\\ &\quad + \mathbf{P}(v_0 \to v_1)\, \mathbf{P}(v_1 \overset{c}{\to} v_n)\\ &= \mathbf{P}(v_0 \to v_n)\, \mathbf{P}(v_1 \not\to v_n, \ldots, v_{n-1} \not\to v_n)\\ &\quad + \sum_{j=1}^{n-2} \mathbf{P}(v_0 \to v_{n-j})\, \mathbf{P}(v_1 \not\to v_{n-j}, \ldots, v_{n-j-1} \not\to v_{n-j}, v_{n-j} \overset{c}{\to} v_n)\\ &\quad + \mathbf{P}(v_0 \to v_1)\, \mathbf{P}(v_1 \overset{c}{\to} v_n).\end{aligned}$$

Using (1), the result follows. □

Lemma 15. *For any* $n \geq 1$, $F_n(p) \geq G_n(p)$ *for all* $p \in [0, 1]$.

Proof. Notice that for the functions G_n inductively defined by

$$G_n(p) = p^n(1-p)^{n-1} + \sum_{j=1}^{n-1} p^{n-j}\, G_j(p)\, (1-p)^{n-j-1}, \quad n \geq 1,$$

we have that G_n bounds F_n from below. This fact can be proved by induction on n, using that $1 - p^k \geq 1 - p$ for all $k \geq 1$. In addition, since

$$G_{n+1}(p) = p(2-p)G_n(p) \quad \text{for every } n \geq 1,$$

we conclude that G_n is given by (5). □

To finish the proof of Theorem 9, we consider n large and find values of p for which the process $(Z_j^n)_{j\geq 0}$ survives with positive probability. We claim that

$$\liminf_{j\to\infty} \mu_j^n > 1 \tag{6}$$

is a sufficient condition for this to happen. Our claim is a direct consequence of Theorem 1 of D'Souza and Biggins (1992), which states that a Galton–Watson process in a varying environment survives with positive probability if it is uniformly supercritical (see the definition in D'Souza and Biggins, 1992) and the quotient between the offspring number of an individual of the jth generation and its mean is dominated by an integrable random variable. As noted in D'Souza and Biggins (1992), (6) implies that the process $(Z_j^n)_{j\geq 0}$ is uniformly supercritical and in our case the required dominating random variable is just the constant $1/F_n(p)$.

To prove Theorem 9, it is enough to assume that $\dim\inf \partial T > 0$. Given $0 < \alpha < \dim\inf \partial T$, there exists $N = N(\alpha)$ such that for any $n \geq N$,

$$\min_{v\in\mathcal{V}} \frac{1}{n} \log M_n(v) > \alpha,$$

therefore $M_n(v) \geq e^{\alpha n}$ for all $v \in \mathcal{V}$ and $n \geq N$. From inequality (4), it follows that for any $n \geq N$,

$$\liminf_{j\to\infty} \mu_j^n \geq e^{\alpha n}\, G_n(p).$$

Now we define

$$\begin{aligned} g_n(p) &= [G_n(p)]^{1/n} - e^{-\alpha}, \quad n \geq N, \\ g(p) &= p(2-p) - e^{-\alpha}, \end{aligned} \tag{7}$$

and note that if p is such that $g_n(p) > 0$, then $\mathcal{M}(T, p)$ survives with positive probability. It is not difficult to show that $\{g_n\}_{n\geq N}$ and g given in (7) satisfy the conditions of the following lemma of Real Analysis, which is stated without proof.

Lemma 16. *Let $\{g_n\}$ be a sequence of increasing, continuous real-valued functions defined on $[0, 1]$, such that $g_n(0) < 0$ and $g_n(1) > 0$ for all n. Suppose that $\{g_n\}$ converges pointwise as $n \to \infty$ to an increasing, continuous function g on $[0, 1]$ and let r_n be the unique root of g_n in $[0, 1]$. Then, there exist $r = \lim_{n\to\infty} r_n$ and $g(r) = 0$.*

Hence, if we define r_n as the unique root of g_n in $[0, 1]$, it follows by Lemma 16 that there exist $r = \lim_{n\to\infty} r_n$ and $g(r) = 0$. Thus, r is the unique root of g in $[0, 1]$. Since $g_n(p) > 0$ for $p > r_n$, we have that $p_c(T) \leq r_n$ for all $n \geq N$. Taking $n \to \infty$, we obtain that

$$p_c(T) \leq r = 1 - (1 - e^{-\alpha})^{1/2}.$$

As this inequality holds for any $0 < \alpha < \dim\inf \partial T$, the desired result is established. □

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